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超幾何分布の分布関数のWiseの近似式の延長

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まえがき

超幾何分布の分布関数を求めたいとき，標本の大きさが大きくなると，その計算はかなりわずらわしくなるので，その近似式をつくっておきたい。

筆者[1],[3]（1953）はロットの大きさNとしたときのN^{-1}の級数展開によって，抜取検査の設計のための計算方法を求めておいたが，M. E. Wise[2],[3]（1954）は（標本の大きさnとし）$M=N-(n-1)/2$としてM^{-2}の級数展開によって近似式を求めることを試みた。Wiseは第2項までを示しているが，第3項を用いるとさらによくなるであろうと思って，求めておいた。

Wiseは巧みな変数変換後の境界積分によって求めている。筆者は$\ln(N!/(N-n)!)$のM^{-1}の $\boxed{\text{級数}}\boxed{\text{漸近}}$ を求め，それをつかって計算を進

めた。

1. $\ln(N!/(N-n)!)$ の逐漸近展開

$M=N-(n-1)/2$ とし、ガンマ関数の対数のStirlingの級数展開を用いて、一般的な形式の漸近展開が得られた。B_{2t-1} はBernoulli数である。

$$
\begin{aligned}
\ln(N!/(N-n)!) &= \ln\Gamma(N+1)-\ln\Gamma(N+1-n)\\
&= \ln\Gamma(M+(n+1)/2)-\ln\Gamma(M-(n-1)/2)\\
&= (M+n/2)\ln(M+(n+1)/2)-(M-n/2)\ln(M-(n-1)/2)\\
&\quad -(M+(n+1)/2)+(M-(n-1)/2)\\
&\quad +\sum_{t=1}(-1)^{t-1}\frac{B_{2t-1}}{2t(2t-1)}\left[\left(M+\frac{n+1}{2}\right)^{-2t+1}-\left(M-\frac{n-1}{2}\right)^{-2t+1}\right]\\
&= n\ln M\\
&\quad -\sum_{\tau=1}^{\infty}\frac{1}{(2M)^{2\tau}}\left[n^{2\tau+1}\frac{1}{2\tau(2\tau+1)}-\sum_{s=1}^{\tau}n^{2\tau+1-2s}\binom{2\tau-1}{2s-2}\left\{\frac{1}{2s}+\sum_{t=1}^{s}(-1)^{t}\frac{B_{2t-1}}{2t(2t-1)}2^{2t}\binom{2s-2}{2t-2}\right\}\right]\\
&\quad -\sum_{\tau=1}^{\infty}\frac{1}{(2M)^{2\tau+1}}\sum_{s=1}^{\tau}n^{2\tau+1-2s}\binom{2\tau}{2s-1}\left\{\frac{1}{2s+1}+\sum_{t=1}^{s}(-1)^{t}\frac{B_{2t-1}}{2t(2t-1)}2^{2t}\binom{2s-1}{2t-2}\right\}
\end{aligned}
\qquad \cdots\cdots (1.1)
$$

一方 $y\,\mathrm{cosech}\,y=[2y/(e^{2y}-1)]\cdot e^{y}$ によって

$$
1+\sum_{s=1}^{\infty}(-1)^{s}B_{2s-1}\frac{y^{2s}\,2(2^{2s-1}-1)}{(2s)!}
$$

$$
=\left\{1-\frac{2y}{2}+\sum_{t=1}^{\infty}(-1)^{t-1}B_{2t-1}\frac{(2y)^{2t}}{(2t)!}\right\}\cdot\left\{1+\sum_{t=1}^{\infty}\frac{y^{t}}{t!}\right\}
$$

$$= 1 + \frac{y}{1!}(1-1) + \sum_{s=2}^{\infty} \frac{y^s}{s!}\left[1 - \binom{s}{1} + \sum_{t=1}^{[s/2]} (-1)^{t-1} B_{2t-1}\, 2^{2t}\binom{s}{2t}\right]$$

であるから，

$$\frac{y^{2s+1}}{(2s+1)!} \text{ の係数} = 0 = 1 - \binom{2s+1}{1} + \sum_{t=1}^{s} (-1)^{t-1} B_{2t-1}\, 2^{2t}\binom{2s+1}{2t}$$

$$\therefore \quad \frac{1}{2s+1} + \sum_{t=1}^{s} (-1)^t \frac{B_{2t-1}}{2t(2t-1)}\, 2^{2t}\binom{2s-1}{2t-2} = 0$$

$$\frac{y^{2s}}{(2s)!} \text{ の係数} = (-1)^s B_{2s-1}\, 2(2^{2s-1}-1)$$

$$= 1 - \binom{2s}{1} + \sum_{t=1}^{s} (-1)^{t-1} B_{2t-1}\, 2^{2t}\binom{2s}{2t}$$

$$= -2s(2s-1)\left\{\frac{1}{2s} + \sum_{t=1}^{s} (-1)^t \frac{B_{2t-1}}{2t(2t-1)}\, 2^{2t}\binom{2s-2}{2t-2}\right\}$$

$$\therefore \quad -\left\{\frac{1}{2s} + \sum_{t=1}^{s} (-1)^t \frac{B_{2t-1}}{2t(2t-1)}\, 2^{2t}\binom{2s-2}{2t-2}\right\}$$

$$= (-1)^s \frac{B_{2s-1}}{2s(2s-1)}\, 2(2^{2s-1}-1)$$

これらを (1・1) につかって

$$\ln(N!/(N-n)!) = n \ln M$$

$$- \sum_{\tau=1}^{\infty} \frac{1}{(2M)^{2\tau}}\left\{\frac{n^{2\tau+1}}{2\tau(2\tau+1)} + \sum_{s=1}^{\tau} n^{2\tau+1-2s}\binom{2\tau-1}{2s-2}(-1)^s \frac{B_{2s-1}}{2s(2s-1)}\, 2(2^{2s-1}-1)\right\}$$

$$\cdots (1\cdot 2)$$

これによって $(2M)^{-2\tau}$ ののぞむ項まで書き下せる。τの小さい所を書けば，次の通り。

$\tau=1$: $\quad n(n^2-1)/(2\cdot3)=(n^{(3)}+3n^{(2)})/6$

$\tau=2$: $\quad (3n^5-10n^3+7n)/(2^2\cdot3\cdot5)$

$$=(3n^{(5)}+30n^{(4)}+65n^{(3)}+15n^{(2)})/60$$

$$=n(n^2-1)(3n^2-7)/60$$

$\tau=3$: $\quad (3n^7-21n^5+49n^3-31n)/(2\cdot3^2\cdot7)$

$$=(3n^{(7)}+63n^{(6)}+399n^{(5)}+840n^{(4)}+427n^{(3)}+21n^{(2)})/126$$

$$=n(n^2-1)(3n^4-18n^2+31)/126$$

$\tau=4$: $\quad (5n^9-60n^7+294n^5-620n^3+381n)/(2^3 3^2 5)$

$$=(5n^{(9)}+180n^{(8)}+2250n^{(7)}+11970n^{(6)}+26649n^{(5)}$$

$$+20790n^{(4)}+3795n^{(3)}+45n^{(2)})/360$$

$$=n(n^2-1)(5n^6-55n^4+239n^2-381)/360$$

$\tau=5$: $\quad (3n^{11}-55n^9+462n^7-2046n^5+4191n^3-2555n)/(2.3.5.11)$

$$=(3n^{(11)}+165n^{(10)}+3410n^{(9)}+33660n^{(8)}+167013n^{(7)}+402633n^{(6)}$$

$$+420519n^{(5)}+151140n^{(4)}+11231n^{(3)}+33n^{(2)})/330$$

$$=n(n^2-1)(3n^8-52n^6+410n^4-1636n^2+2555)/330$$

ここで $n^{(m)}=n(n-1)(n-2)\cdots(n-m+1)$ である.

2. 計算の準備

ロットの大きさN, 標本の大きさn, 母集団不良率p_0, 標本のなかの不良品個数がcを超えない確率Pはよく知られている通り

$$P=\sum_{d=0}^{c}\frac{n!\,(N-n)!\,(Np_0)!\,(N-Np_0)!}{d!\,(n-d)!\,N!\,(Np_0-d)!\,(N-Np_0-n+d)!}\qquad\cdots(2.1)$$

$\ln(N!/(N-n)!)$ は $M=N-(n-1)/2$ を用いて，(1.2)によって $(2M)^{-2\tau}$ の漸近級数に展開できることを示しておいたが，(2.1)には $(Np_0)!/(Np_0-d)!$，$(N-Np_0)!/(N-Np_0-n+d)!$ があるから，これも同じように(1.2)によって漸近級数に展開できるであろう．その上で P の初項が標本の大きさ n，~~母~~集団不良率 p で，標本のなかの不良品箇が c を超えない二項分布の確率であらわされ，$(2M)^{-1}$ の項が0となるように p を決定する。

$$M_p=Np_0-\frac{d-1}{2},\quad M_q=Nq_0-\frac{n-d-1}{2}\qquad\cdots(2.2)$$

$$p_0=p(1+x/(2M)),\quad 1-p_0=q_0=q-px/(2M)\qquad\cdots(2.3)$$

とおくと

$$M_p=M\left(1+\frac{n-1}{2M}\right)p\left(1+\frac{x}{2M}\right)-\frac{d-1}{2}$$

$$=Mp\left\{1+\frac{n-1-(d-1)p^{-1}}{2M}+\frac{x}{2M}\left(1+\frac{n-1}{2M}\right)\right\}\qquad\cdots(2.4p)$$

$$M_q=Mq\left\{1+\frac{n-1-(n-d-1)q^{-1}}{2M}-\frac{p}{q}\frac{x}{2M}\left(1+\frac{n-1}{2M}\right)\right\}\qquad\cdots(2.4q)$$

ここで

$$x_1 = x\,(1 + (n-1)/(2M)) \quad \cdots (2.5)$$

とおくと，

$$M_p = Mp\{1 + [n-1-(d-1)p^{-1} + x_1]/(2M)\} \quad \cdots (2.6p)$$

$$M_q = Mq\{1 + [n-1-(n-d-1)q^{-1} - x_1 p q^{-1}]/2M\} \quad \cdots (2.6q)$$

(1.2) 式の右辺才1項によって

$$-n \ln M + d \ln M_p + (n-d) \ln M_q$$

$$= d \ln p + (n-d) \ln q + d \ln\{1 + [n-1-(d-1)p^{-1} + x_1]/(2M)\}$$

$$+ (n-d) \ln\{1 + [n-1-(n-d-1)q^{-1} - x_1 p q^{-1}]/(2M)\}$$

$$\cdots (2.7)$$

(1.2) 式の右辺才2項は $(2M)^{-2}$ ではじまるから，(2.1) で $(2M)^{-1}$ の係数に対しては (2.7) からのみである．(2.7) によって

$$\frac{1}{2M}\text{の係数} = \sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d} \{ d^{(1)} [n-1-(d-1)p^{-1} + x_1]$$

$$+ (n-d)^{(1)} [n-1-(n-d-1)q^{-1} - x_1 p q^{-1}]\}$$

$$= n^{(1)}\{(n-1+x_1)\,p\,B(c-1, n-1, p) + (n-1-\frac{x_1 p}{q})\,q\,B(c, n-1, p)\}$$

$$- n^{(2)}\{p\,B(c-2, n-2, p) + q\,B(c, n-2, p)\} \quad \cdots (2.8)$$

ここに $B(c, n, p) = \sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d}$ である．なお

$$b(d, n, p) = \binom{n}{d} p^d q^{n-d}$$

とし，特にまぎらわしくないとき $B(c, n)$, $b(d, n)$ のように記す．

(2.8) を整理すると

$$\frac{1}{2M} \text{の係数} = n^{(1)}\{[(n-1)q - x_1 p]\, b(c, n-1, p)$$
$$+ ((n-1+x_1)p + [(n-1)q - x_1 p])\, B(c-1, n-1, p)\}$$
$$- n^{(2)}\{p\, B(c-2, n-2, p) + q\, B(c, n-2, p)\}$$
$$= n\, b(c, n-1, p)\, [(n-1)q - x_1 p]$$
$$+ n^{(2)}\{B(c-1, n-1, p) - p\, B(c-2, n-2, p) - q\, B(c, n-2, p)\}$$
$$= n\, b(c, n-1, p)[(n-1)q - x_1 p] - n^{(2)} q\, b(c, n-2, p)$$
$$= n\, b(c, n-1, p)[(n-1)q - x_1 p - (n-1-c)] = 0$$

$$\therefore \quad x_1 p = (n-1)q - (n-1-c) = cq - (n-1-c)p \quad \cdots (2.9)$$

$$\left.\begin{aligned} n-1+x_1 &= n-1+\frac{cq-(n-1-c)p}{p} = \frac{c}{p} \\ n-1-\frac{x_1 p}{q} &= n-1-\frac{cq-(n-1-c)p}{q} = \frac{n-1-c}{q} \end{aligned}\right\} \cdots (2.10)$$

なお次の関係をつかっている．

$$B(c, n, p) = \sum_{d=0}^{c} \binom{n-1}{d} p^d q^{n-d} + \sum_{d=0}^{c-1} \binom{n-1}{d} p^{d+1} q^{n-1-d}$$
$$= B(c, n-1, p)\cdot q + B(c-1, n-1, p)\cdot p$$
$$= \sum_{i=0}^{\lambda} B(c-i, n-\lambda, p)\binom{\lambda}{i} p^i q^{\lambda-i} \quad \cdots (2.11)$$

(2.10) によって、(2.6) は

$$\left.\begin{aligned} M_p &= Mp\left(1-\frac{d-1-c}{2Mp}\right) \\ M_q &= Mq\left(1-\frac{n-d-1-(n-1-c)}{2Mq}\right) \end{aligned}\right\} \cdots (2.12)$$

(2.3), (2.5) によって

$$p_0 = p\left\{1+\frac{x_1}{2M}\left(1+\frac{n-1}{2M}\right)^{-1}\right\}$$

$$= p + \frac{cq-(n-1-c)p}{2N} = \left(pM+\frac{c}{2}\right)N^{-1} \quad \cdots (2.13)$$

$$Mp = Np_0 - \frac{c}{2}, \quad Mq = Np_0 - \frac{n-1-c}{2} \quad \cdots\cdots (2.14)$$

Wise が h, 1−x, p としているのは、それぞれここでの q, q_0, c+1 である。

(2.7) によって

$$\left(1-\frac{d-1-c}{2Mp}\right)^d\left(1-\frac{n-d-1-(n-1-c)}{2Mq}\right)^{n-d}$$

$$= \left\{1+\sum_{\lambda=1}^{d}(-1)^\lambda\frac{d^{(\lambda)}}{\lambda!}\left(\frac{d-1-c}{2Mp}\right)^\lambda\right\}\cdot\left\{1+\sum_{\lambda=1}^{n-d}(-1)^\lambda\frac{(n-d)^{(\lambda)}}{\lambda!}\left(\frac{n-d-1-(n-1-c)}{2Mq}\right)^\lambda\right\}$$

$$= 1+\sum_{\lambda=1}^{n}(-1)^\lambda\frac{1}{\lambda!(2M)^\lambda}G_\lambda \quad \cdots\cdots (2.15.1)$$

$$G_\lambda = \sum_{i=0}^{\lambda}\binom{\lambda}{i}d^{(\lambda-i)}(n-d)^{(i)}\left(\frac{d-1-c}{p}\right)^{\lambda-i}\left(\frac{n-d-1-(n-1-c)}{q}\right)^i$$

($G_0=1$ とおく) $\cdots\cdots$ (2.15.2)

(1.2) の右辺第2項を書き直し

$$- \sum_{\tau=1}^{\infty} \frac{1}{(2\tau)!\,(2M)^{2\tau}} f_{2\tau}(n)$$

と書くと，(2.1)は次のようになる．(なお $f_{2\tau}(n)$ は附録1のようになる)

$$P = \sum_{d=0}^{c} \binom{n}{d} p^{d} q^{n-d} \left\{ 1 + \sum_{\lambda=1}^{n} (-1)^{\lambda} \frac{1}{\lambda!\,(2M)^{\lambda}} G_{\lambda} \right\} \cdot$$

$$\cdot \exp\Big[\sum_{\tau=1}^{\infty} \frac{1}{(2\tau)!\,(2M)^{2\tau}} \Big\{ f_{2\tau}(n) - \frac{1}{p^{2\tau}} f_{2\tau}(d) \Big(1 - \frac{d-1-c}{2Mp}\Big)^{-2\tau}$$

$$- \frac{1}{q^{2\tau}} f_{2\tau}(n-d) \Big(1 - \frac{n-d-1-(n-1-c)}{2Mq}\Big)^{-2\tau} \Big\} \Big]$$

$$\cdots\cdots (2.16)$$

(2.16)を整理するため

$$\phi_m = \sum_{\tau=1}^{[\frac{m}{2}]} m^{(m-2\tau)} \binom{m-1}{m-2\tau} \Big\{ \frac{1}{p^m} f_{2\tau}(d) \cdot (d-1-c)^{m-2\tau}$$

$$m \geq 2 \qquad\qquad + \frac{1}{q^m} f_{2\tau}(n-d) \cdot (n-d-1-(n-1-c))^{m-2\tau} \Big\}$$

$$\cdots (2.17)$$

$$F_{2m} = \sum_{\alpha=1}^{m} \sum_{2m}^{\alpha} \frac{(2m)!}{\alpha_2!\,(2!)^{\alpha_2}\,\alpha_4!\,(4!)^{\alpha_4}\cdots} (f_2(n))^{\alpha_2} (f_4(n))^{\alpha_4} \cdots$$

$$\cdots\cdots (2.18)$$

$\sum_{2m}^{\alpha}$ は $\left\{ \begin{array}{l} \alpha = \alpha_2 + \alpha_4 + \cdots\cdots \qquad ; \alpha_2, \alpha_4, \cdots \geq 0 \\ 2m = 2\alpha_2 + 4\alpha_4 + \cdots \end{array} \right\}$

である $\alpha_2, \alpha_4, \cdots$ のあらゆる組合せの総和の意味

$$\Phi_m = \sum_{\alpha=1}^{[\frac{m}{2}]} (-1)^{\alpha} \sum_{m}^{\alpha} \frac{m!}{\alpha_2!\,(2!)^{\alpha_2}\,\alpha_3!\,(3!)^{\alpha_3}\,\alpha_4!\,(4!)^{\alpha_4}\cdots} \cdot$$

$$\cdot \phi_2^{\alpha_2} \phi_3^{\alpha_3} \phi_4^{\alpha_4} \cdots\cdots \qquad \cdots (2.19)$$

$\sum\limits_{m}^{\alpha}$ は $\left\{ \begin{array}{l} \alpha = \alpha_2 + \alpha_3 + \alpha_4 + \cdots \\ m = 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \cdots \end{array} \quad ; \ \alpha_2, \alpha_3, \alpha_4, \cdots \geq 0 \right\}$

である $\alpha_2, \alpha_3, \alpha_4, \cdots$ のあらゆる組合せの総和.

F_{2m} の値は付録 2のようになる.

(2.17), (2.18), (2.19) を用いると (2.16) は

$$P = \sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{\lambda=1}^{n} (-1)^\lambda \frac{1}{\lambda!(2M)^\lambda} G_\lambda \right\} \cdot$$

$$\cdot \left\{ 1 + \sum_{m=1}^{\infty} \left[\frac{1}{(2m)!(2M)^{2m}} \sum_{s=0}^{m} \frac{(2m)!}{(2s)!(2m-2s)!} F_{2m-2s} \Phi_{2s} \right.\right.$$

$$\left.\left. + \frac{1}{(2m+1)!(2M)^{2m+1}} \sum_{s=1}^{m} \frac{(2m+1)!}{(2s+1)!(2m-2s)!} F_{2m-2s} \Phi_{2s+1} \right] \right\}$$

$$= \sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d} \left\{ 1 + \sum_{m=1}^{\infty} \frac{1}{(2m)!(2M)^{2m}} \left[F_{2m} \right.\right.$$

$$+ \sum_{t=1}^{m} F_{2m-2t} \binom{2m}{2t} \left(\sum_{s=0}^{t} \binom{2t}{2s} \Phi_{2s} G_{2t-2s} \right.$$

$$\left.\left. - \sum_{s=1}^{t-1} \binom{2t}{2s+1} \Phi_{2s+1} G_{2t-1-2s} \right) \right]$$

$$+ \sum_{m=1}^{\infty} \frac{1}{(2m+1)!(2M)^{2m+1}} \left[-\binom{2m+1}{1} F_{2m} G_1 \right.$$

$$+ \sum_{t=1}^{m} F_{2m-2t} \binom{2m+1}{2t+1} \left(\sum_{s=1}^{t} \binom{2t+1}{2s+1} \Phi_{2s+1} G_{2t-2s} \right.$$

$$\left.\left.\left. - \sum_{s=0}^{t} \binom{2t+1}{2s} \Phi_{2s} G_{2t+1-2s} \right) \right] \right\}$$

$$\cdots\cdots (2.10)$$

{ }の

となる. (2.10) の示すことは $(p, q), (d, n-d), (c, \bar{c})$ について対.構造をもつことである.

3. $(2M)^{-(2m+1)}$ の項について

(2.10) の $(2M)^{-(2m+1)}$ の項について, これが 0 となることを, m の小さいときに計算した.

まず係数の第1項は

$$\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\,G_1$$

$$=\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\left[\frac{d(d-1-c)}{p}+\frac{(n-d)(n-d-1-(n-1-c))}{q}\right]$$

$$=n^{(2)}\sum_{d=0}^{c-2}\binom{n-2}{d}p^{d+1}q^{n-2-d}-n^{(1)}\sum_{d=0}^{c-1}\binom{n-1}{d}p^d q^{n-1-d}\cdot c$$

$$+n^{(2)}\sum_{d=0}^{c}\binom{n-2}{d}p^d q^{n-1-d}-n^{(1)}\sum_{d=0}^{c}\binom{n-1}{d}p^d q^{n-1-d}\cdot(n-1-c)$$

$$=n^{(2)}B(c-2,n-2)\,p+n^{(2)}B(c,n-2)\,q$$

$$-n\,B(c-1,n-1)\;c-n\,B(c,n-1)\;(n-1-c)$$

$$=-n\,b(c,n-1)\cdot(n-1-c)+n^{(2)}\left[B(c-2,n-2)p+B(c,n-2)q\right.$$

$$\left.-B(c-1,n-1)\right]$$

$$=-n\,b(c,n-1)\cdot(n-1-c)+n^{(2)}b(c,n-2)\,q$$

$$=n\,b(c,n-1)\{-(n-1-c)+(n-1-c)\}=0\qquad\cdots(3.1)$$

次に $F_{2m-2t}\binom{2m+1}{2t+1}$ の係数については

$$A_{2t+1}=\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\left[\sum_{s=1}^{t}\binom{2t+1}{2s+1}\Phi_{2s+1}\,G_{2t-2s}\right.$$

$$\left.-\sum_{s=0}^{t}\binom{2t+1}{2s}\Phi_{2s}\,G_{2t+1-2s}\right]$$

$$\cdots(3.2)$$

とおくと

$$A_3=\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\left[\Phi_3-\binom{3}{2}\Phi_2\,G_1-G_3\right]\qquad\cdots(3.3)$$

今便宜上 $\bar{c}=(n-1-c)$ とおき，(p,q)，$(d,n-d)$，

$(c, \bar{c})$ を同時に取り換えたものを加える意味をあらわして $+\sim$ と記しておくと

$$\Phi_3 - 3\Phi_2 G_1 - G_3$$

$$= -6\left\{\frac{1}{p^3} f_2(d)\cdot(d-1-c) + \sim\right\} + 3\left\{\frac{1}{p^2} f_2(d) + \sim\right\}\left\{\frac{d(d-1-c)}{p} + \sim\right\}$$

$$-\left\{\frac{d^{(3)}(d-1-c)^3}{p^3} + 3\frac{d^{(2)}(d-1-c)^2(n-d)(n-d-1-\bar{c})}{p^2 q} + \sim\right\}$$

$$= \frac{1}{p^3}\left[-d^{(6)} + d^{(5)}(c3-8) - d^{(4)}(c^{(2)}3 - c11 + 12) + d^{(3)}(c^{(3)} - c^{(2)}3 + c3)\right]$$

$$+ \frac{1}{p^2 q}\left[-d^{(4)}3 + d^{(3)}(c6-8) - d^{(2)}(c^{(2)}3 - c3)\right]\cdot\left[(n-d)^{(2)} - (n-d)\bar{c}\right]$$

$$+\sim$$

となるから，(3.3) に入れて

$$A_3 = -n^{(6)}B(c, n-6)q^3 + n^{(5)}B(c, n-5)q^2[\bar{c}3-8]$$

$$-n^{(4)}B(c, n-4)q[\bar{c}^{(2)}3 - \bar{c}11 + 12] + n^{(3)}B(c, n-3)[\bar{c}^{(3)} - \bar{c}^{(2)}3 + \bar{c}3]$$

$$-n^{(6)}B(c-2, n-6)pq^2 3 + n^{(5)}B(c-2, n-5)pq[\bar{c}6-8] - n^{(4)}B(c-2, n-4)p\cdot$$
$$\cdot[\bar{c}^{(2)}3 - \bar{c}3]$$

$$+c\left\{n^{(5)}B(c-1, n-5)q^2 3 - n^{(4)}B(c-1, n-4)q[\bar{c}6-8] + n^{(3)}B(c-1, n-3)[\bar{c}^{(2)}3 - \bar{c}3]\right\}$$

$$-n^{(6)}B(c-4, n-6)p^2 q 3 + n^{(5)}B(c-3, n-5)pq[c6-8] - n^{(4)}B(c-2, n-4)q[c^{(2)}3 - c3]$$

$$+\bar{c}\left\{n^{(5)}B(c-4, n-5)p^2 3 - n^{(4)}B(c-3, n-4)p[c6-8] + n^{(3)}B(c-2, n-3)[c^{(2)}3 - c3]\right\}$$

$$-n^{(6)}B(c-6, n-6)p^3 + n^{(5)}B(c-5, n-5)p^2[c3-8]$$

$$-n^{(4)}B(c-4, n-4)p[c^{(2)}3 - c11 + 12] + n^{(3)}B(c-3, n-3)[c^{(3)} - c^{(2)}3 + c3] \quad \cdots (3.4)$$

ここで $B(c, n-6)$ などの項について

$$-n^{(6)}b(c-\lambda, n-6)q^3 + n^{(5)}b(c-\lambda, n-5)q^2[\bar{c}3-8]$$

$$-n^{(4)}b(c-\lambda,n-4)q\,[\bar{c}^{(2)}3-\bar{c}\,11+12)+n^{(3)}b(c-\lambda,n-3)(\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}\,3)$$

$$=nb(c,n-1)\frac{c^{(\lambda)}}{p^{\lambda}q^{2-\lambda}}\Big\{-\bar{c}^{(5-\lambda)}+\bar{c}^{(4-\lambda)}[\bar{c}\,3-8]-\bar{c}^{(3-\lambda)}[\bar{c}^{(2)}3-\bar{c}\,11+12]$$
$$+\bar{c}^{(2-\lambda)}[\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}\,3]\Big\}$$

$$=nb(c,n-1)\frac{c^{(\lambda)}}{p^{\lambda}q^{2-\lambda}}\Big\{\bar{c}^{(3-\lambda)}\binom{\lambda}{1}-\bar{c}^{(2-\lambda)}\Big[\binom{\lambda}{3}6+\binom{\lambda}{2}6-\binom{\lambda}{1}3\Big]\Big\}$$

$(0\leq\lambda\leq 2)$ … (3.5)

$\lambda=0$ のとき 0 となる.

次に

$$-n^{(6)}b(c-1-\lambda,n-6)pq^{2}3+n^{(5)}b(c-1-\lambda,n-5)pq\,[\bar{c}\,6-8]$$
$$-n^{(4)}b(c-1-\lambda,n-4)\,p\,[\bar{c}^{(2)}3-\bar{c}\,3]$$

$$+c\Big\{n^{(5)}b(c-\lambda,n-5)q^{2}3-n^{(4)}b(c-\lambda,n-4)q\,[\bar{c}\,6-8]+n^{(3)}b(c-\lambda,n-3)[\bar{c}^{(2)}3-\bar{c}\,3]\Big\}$$

$$=nb(c,n-1)\frac{-c^{(1+\lambda)}+c\cdot c^{(\lambda)}}{p^{\lambda}q^{2-\lambda}}\Big\{\bar{c}^{(4-\lambda)}3-\bar{c}^{(3-\lambda)}[\bar{c}\,6-8]+\bar{c}^{(2-\lambda)}[\bar{c}^{(2)}3-\bar{c}\,3]\Big\}$$

$$=nb(c,n-1)\frac{c^{(\lambda)}\lambda}{p^{\lambda}q^{2-\lambda}}\Big\{-\bar{c}^{(3-\lambda)}+\bar{c}^{(2-\lambda)}\Big[\binom{\lambda-1}{2}6+\binom{\lambda-1}{1}3-3\Big]\Big\}$$

$(1\leq\lambda\leq 2)$

$$=nb(c,n-1)\frac{c^{(\lambda)}}{p^{\lambda}q^{2-\lambda}}\Big\{-\bar{c}^{(3-\lambda)}\binom{\lambda}{1}+\bar{c}^{(2-\lambda)}\Big[\binom{\lambda}{3}18+\binom{\lambda}{2}6-\binom{\lambda}{1}3\Big]\Big\}$$

… (3.6)

次のこれと対称な項では ~~[illegible]~~ (3.6) を用い

$$nb(c,n-1)\frac{\bar{c}^{(\lambda)}}{p^{2-\lambda}q^{\lambda}}\Big\{-c^{(3-\lambda)}\binom{\lambda}{1}+c^{(2-\lambda)}\Big[\binom{\lambda}{3}18+\binom{\lambda}{2}6-\binom{\lambda}{1}3\Big]\Big\}$$

で $\lambda=0$ のと 0.

(3.5) と (3.6) とを加えると

$$nb(c,n-1)\frac{c^{(\lambda)}}{p^{\lambda}q^{2-\lambda}}\bar{c}^{(2-\lambda)}\binom{\lambda}{3}12=0\ (1\leq\lambda\leq 2)$$

となり，(3.4)は次のようになる。

$$A_3 = -n^{(6)}B(c-3,n-6)q^3+n^{(5)}B(c-3,n-5)q^2[\bar{c}3-8]-n^{(4)}B(c-3,n-4)q[\bar{c}^{(2)}3-\bar{c}11+12]$$

$$+n^{(3)}B(c-3,n-3)[\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}3]$$

$$-n^{(6)}B(c-4,n-6)pq^2 3+n^{(5)}B(c-4,n-5)pq[\bar{c}6-8]-n^{(4)}B(c-4,n-4)p[\bar{c}^{(2)}3-\bar{c}3]$$

$$+c\{n^{(5)}B(c-3,n-5)q^2 3-n^{(4)}B(c-3,n-4)q[\bar{c}6-8]+n^{(3)}B(c-3,n-3)[\bar{c}^{(2)}3-\bar{c}3]\}$$

$$-n^{(6)}B(c-5,n-6)p^2q3+n^{(5)}B(c-4,n-5)pq[c6-8]-n^{(4)}B(c-3,n-4)q[c^{(2)}3-c3]$$

$$+\bar{c}\{n^{(5)}B(c-5,n-5)p^2 3-n^{(4)}B(c-4,n-4)p[c6-8]+n^{(3)}B(c-3,n-3)[c^{(2)}3-c3]\}$$

$$-n^{(6)}B(c-6,n-6)p^3+n^{(5)}B(c-5,n-5)p^2[c3-8]-n^{(4)}B(c-4,n-4)p[c^{(2)}3-c11+12]$$

$$+n^{(3)}B(c-3,n-3)[c^{(3)}-c^{(2)}3+c3]$$

$$=-n^{(6)}B(c-3,n-3)+n^{(5)}B(c-3,n-3)[(n-1)3-8]-n^{(4)}B(c-3,n-3)[(n-1)^{(2)}3-(n-1)11+12]$$

$$+n^{(3)}B(c-3,n-3)[(n-1)^{(3)}-(n-1)^{(2)}3+(n-1)3]$$

$$=B(c-3,n-3)\{-n^{(6)}+n^{(5)}[(n-5)3+4]-n^{(4)}[(n-4)^{(2)}3+(n-4)7-3]$$

$$+n^{(3)}[(n-3)^{(3)}+(n-3)^{(2)}3-(n-3)3]\}$$

$$=0$$

このような方法で $A_5=0$，$A_7=0$ が計算された。

4. $(2M)^{-2m}$ の項について

(2.10) の $1/((2m)!(2M)^{2m})$ の係数を K_{2m} とおると

$$K_{2m}=\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\left[F_{2m}+\sum_{t=1}^{m}F_{2m-2t}\binom{2m}{2t}A_{2t}\right] \quad \cdots(4.1)$$

$$A_{2t}=\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\left\{\sum_{s=0}^{t}\binom{2t}{2s}\Phi_{2s}G_{2t-2s}-\sum_{s=1}^{t-1}\binom{2t}{2s+1}\Phi_{2s+1}G_{2t-1-2s}\right\} \quad \cdots(4.2)$$

(4.1) の F_{2m} の係数は (2.11) によって

$$\sum_{d=0}^{c}\binom{n}{d}p^{d}q^{n-d} = B(c,n)$$

$$= B(c,n-2)q^{2} + B(c-1,n-2)pq^{2}$$

$$+ B(c-2,n-2)p^{2}$$

$$= b(c,n-2)q^{2} - b(c-1,n-2)p^{2} + B(c-1,n-2)$$

$$= \frac{nb(c,n-1)}{n^{(2)}}A_{0,0} + B(c-1,n-2) \quad \cdots\cdots (4.3)$$

$$A_{0,0} = \bar{c}q - cp \quad \cdots\cdots (4.4)$$

同じようにして

$$B(c-\lambda,n-2\lambda) = B(c-\lambda,n-2\lambda-2)q^{2}$$

$$+ B(c-\lambda-1,n-2\lambda-2)pq2 + B(c-\lambda-2,n-2\lambda-2)p^{2}$$

$$= b(c-\lambda,n-2\lambda-2)q^{2} - b(c-\lambda-1,n-2\lambda-2)p^{2}$$

$$+ B(c-\lambda-1,n-2\lambda-2)$$

$$= \frac{nb(c,n-1)}{n^{(2\lambda+2)}}A_{0,\lambda} + B(c-\lambda-1,n-2\lambda-2) \quad \cdots (4.5)$$

$$A_{0,\lambda} = \frac{c^{(\lambda)}\bar{c}^{(\lambda+1)}}{p^{\lambda}q^{\lambda-1}} - \frac{c^{(\lambda+1)}\bar{c}^{(\lambda)}}{p^{\lambda-1}q^{\lambda}} \quad \cdots (4.6)$$

次に A_2 を計算すると

$$A_2 = \cancel{\Phi_2 + G_2} \sum_{d=0}^{c}\binom{n}{d}p^{d}q^{n-d}\{\Phi_2 + G_2\}$$

$$= \sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d} \Big\{ - \frac{d^{(3)} + d^{(2)}3}{3p^2} - \sim$$

$$+ \frac{d^{(4)} - d^{(3)}(c2-3) + d^{(2)}(c^{(2)} - c + 1)}{p^2} + \sim$$

$$+ \frac{2}{pq}\left(d^{(2)} - dc\right)\left((n-d)^{(2)} - (n-d)\bar{c}\right)\Big\}$$

$$= \sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d} \Big\{ \frac{1}{p^2}\left[d^{(4)} - d^{(3)}(c2 - \tfrac{8}{3}) + d^{(2)}(c^{(2)} - c)\right] + \sim$$

$$+ \frac{2}{pq}\left(d^{(2)} - dc\right)\left((n-d)^{(2)} - (n-d)\bar{c}\right)\Big\}$$

$$= n^{(4)}B(c, n-4)\,q^2 - n^{(3)}B(c, n-3)\,q\,(\bar{c}2 - \tfrac{8}{3}) + n^{(2)}B(c, n-2)(\bar{c}^{(2)} - \bar{c})$$

$$+ n^{(4)}B(c-4, n-4)\,p^2 - n^{(3)}B(c-3, n-3)\,p\,(c2 - \tfrac{8}{3}) + n^{(2)}B(c-2, n-2)(c^{(2)} - c)$$

$$+ n^{(4)}B(c-2, n-4)\,pq\,2 - n^{(3)}B(c-1, n-3)\,q\,c2$$

$$- n^{(3)}B(c-2, n-3)\,p\,\bar{c}\,2 + n^{(2)}B(c-1, n-2)\,c\bar{c}\,2$$

$$= n^{(4)}\left[b(c, n-4)q^2 - b(c-3, n-4)p^2\right] + n^{(4)}\left[B(c-1, n-4)q^2\right.$$

$$\left. + B(c-2, n-4)\,pq\,2 + B(c-3, n-4)\,p^2\right]$$

$$- n^{(3)}\left[b(c, n-3)\,q\,(\bar{c}2 - \tfrac{8}{3}) - b(c-2, n-3)\,p\,(c2 - \tfrac{8}{3})\right]$$

$$- n^{(3)}\left[B(c-1, n-3)\,q\,((n-1)2 - \tfrac{8}{3}) + B(c-2, n-3)\,p\,((n-1)2 - \tfrac{8}{3})\right.$$

$$+ n^{(2)}\left[b(c, n-2)(\bar{c}^{(2)} - \bar{c}) - b(c-1, n-2)(c^{(2)} - c)\right]$$

$$+ n^{(2)}B(c-1, n-2)\left((n-1)^{(2)} - (n-1)\right)$$

$$= nb(c, n-1)A_{2,0} + B(c-1, n-2)\Big\{ n^{(4)} - n^{(3)}\left((n-3)2 + \tfrac{4}{3}\right)$$

$$+ n^{(2)}\left((n-2)^{(2)} + (n-2) - 1\right)\Big\}$$

$$= nb(c, n-1)A_{2,0} - B(c-1, n-2)\,f_2(n) \quad \ldots \quad (4.7)$$

$$A_{2,0} = \frac{\bar{c}^{(3)}}{q} - \frac{c^{(3)}}{p} - \frac{\bar{c}^{(2)}}{q}\left(\bar{c}\,2 - \frac{8}{3}\right) + \frac{c^{(2)}}{p}\left(c\,2 - \frac{8}{3}\right)$$

$$+ \frac{\bar{c}}{q}\left(\bar{c}^{(2)} - \bar{c}\right) - \frac{c}{p}\left(c^{(2)} - c\right)$$

$$= - \frac{\bar{c}(\bar{c}+2)}{3q} + \frac{c(c+2)}{3p} \qquad \cdots\cdots \quad (4.8)$$

よって (4.1) によって

$$K_{2m} = nb(c, n-1)\left\{ \frac{F_{2m}}{n^{(2)}} A_{0,0} + \frac{F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n)}{n^{(4)}} A_{0,1} + \binom{2m}{2} F_{2m-2} A_{2,0} \right\}$$

$$+ \left[F_{2m} - \binom{2m}{2} F_{2m-2} f_2(n) \right] B(c-2, n-4)$$

$$\cancel{\sum_{d=0}} \qquad + \sum_{t=2}^{m} F_{2m-2t} \binom{2m}{2t} A_{2t} \qquad \cdots\cdots \quad (4.9)$$

すなわち

$$K_2 = nb(c, n-1)\left\{ \frac{n+1}{3} A_{0,0} + A_{2,0} \right\} \qquad \cdots \quad (4.10)$$

となる。

次に A_4 について計算する。

$$A_4 = \sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d} \left\{ \sum_{s=0}^{2} \binom{4}{2s} \Phi_{2s} G_{4-2s} - \binom{4}{1} \Phi_3 G_1 \right\}$$

$$\cdots \quad (4.11)$$

まず (4.11) の { } 内の d の次の項，$(n-d)$ の次の項を集め

$$\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\left\{\frac{1}{p^4}\Big[d^{(8)}-d^{(7)}(c4-16)+d^{(6)}\left(c^{(2)}6-c38+\frac{208}{3}\right)\right.$$
$$-d^{(5)}\left(c^{(3)}4-c^{(2)}28+c76-\frac{384}{5}\right)$$
$$\left.+d^{(4)}\left(c^{(4)}-c^{(3)}6+c^{(2)}15-c15\right)\Big]+\sim\right\}$$

$$= n^{(8)}B(c-8,n-8)p^4-n^{(7)}B(c-7,n-7)p^3(c4-16)$$
$$+n^{(6)}B(c-6,n-6)p^2\left(c^{(2)}6-c38+\frac{208}{3}\right)-n^{(5)}B(c-5,n-6)p\left(c^{(3)}4-c^{(2)}28+c76-\frac{384}{5}\right)$$
$$+n^{(4)}B(c-4,n-4)\left(c^{(4)}-c^{(3)}6+c^{(2)}15-c15\right)$$
$$+n^{(8)}B(c,n-8)q^4-n^{(7)}B(c,n-7)q^3(\bar{c}4-16)$$
$$+n^{(6)}B(c,n-6)q^2\left(\bar{c}^{(2)}6-\bar{c}38+\frac{208}{3}\right)-n^{(5)}B(c,n-6)q\left(\bar{c}^{(3)}4-\bar{c}^{(2)}28+\bar{c}76-\frac{384}{5}\right)$$
$$+n^{(4)}B(c,n-4)\left(\bar{c}^{(4)}-\bar{c}^{(3)}6+\bar{c}^{(2)}15-\bar{c}15\right)$$

$$=\sum_{\lambda=0}^{1}\Big\{n^{(8)}b(c-\lambda,n-8)q^4-n^{(7)}b(c-\lambda,n-7)q^3(\bar{c}4-16)+n^{(6)}b(c-\lambda,n-6)q^2\left(\bar{c}^{(2)}6-\bar{c}38+\frac{208}{3}\right)$$
$$-n^{(5)}b(c-\lambda,n-5)q\left(\bar{c}^{(3)}4-\bar{c}^{(2)}28+\bar{c}76-\frac{384}{5}\right)+n^{(4)}b(c-\lambda,n-4)\left(\bar{c}^{(4)}-\bar{c}^{(3)}6+\bar{c}^{(2)}15-\bar{c}15\right)$$
$$-n^{(8)}b(c-7+\lambda,n-8)p^4+n^{(7)}b(c-6+\lambda,n-7)p^3(c4-16)+n^{(6)}b(c-5+\lambda,n-6)p^2\left(c^{(2)}6-c38+\frac{208}{3}\right)$$
$$+n^{(5)}b(c-4+\lambda,n-5)p\left(c^{(3)}4-c^{(2)}28+c76-\frac{384}{5}\right)-n^{(4)}b(c-3+\lambda,n-4)\left(c^{(4)}-c^{(3)}6+c^{(2)}15-c15\right)\Big\}$$
$$+n^{(8)}B(c-2,n-8)q^4-n^{(7)}B(c-2,n-7)q^3(\bar{c}4-16)+n^{(6)}B(c-2,n-6)q^2\left(\bar{c}^{(2)}6-\bar{c}38+\frac{208}{3}\right)$$
$$-n^{(5)}B(c-2,n-5)q\left(\bar{c}^{(3)}4-\bar{c}^{(2)}28+\bar{c}76-\frac{384}{5}\right)+n^{(4)}B(c-2,n-4)\left(\bar{c}^{(4)}-\bar{c}^{(3)}6+\bar{c}^{(2)}15-\bar{c}15\right)$$
$$+n^{(8)}B(c-6,n-8)p^4-n^{(7)}B(c-5,n-7)p^3(c4-16)+n^{(6)}B(c-4,n-6)p^2\left(c^{(2)}6-c38+\frac{208}{3}\right)$$
$$-n^{(5)}B(c-3,n-5)p\left(c^{(3)}4-c^{(2)}28+c76-\frac{384}{5}\right)+n^{(4)}B(c-2,n-4)\left(c^{(4)}-c^{(3)}6+c^{(2)}15-c15\right)$$

$$=nb(c,n-1)\sum_{\lambda=0}^{1}\left\{\frac{c^{(\lambda)}}{p^\lambda q^{3-\lambda}}\Big[\bar{c}^{(7-\lambda)}-\bar{c}^{(6-\lambda)}(\bar{c}4-16)+\bar{c}^{(5-\lambda)}\left(\bar{c}^{(2)}6-\bar{c}38+\frac{208}{3}\right)\right.$$
$$-\bar{c}^{(4-\lambda)}\left(\bar{c}^{(3)}4-\bar{c}^{(2)}28+\bar{c}76-\frac{384}{5}\right)$$
$$+\bar{c}^{(3-\lambda)}\left(\bar{c}^{(4)}-\bar{c}^{(3)}6-\bar{c}^{(2)}15+\bar{c}15\right)\Big]$$
$$\left.-\sim\right\}+n^{(8)}B(c-2,n-8)q^4-\cdots\cdots$$

$$= n\, b(c, n-1) \sum_{\lambda=0}^{1} \left\{ \frac{c^{(\lambda)}}{p^{\lambda} q^{3-\lambda}} \left[\bar{c}^{(5-\lambda)} \frac{1}{3} + \bar{c}^{(4-\lambda)} \left[-\binom{\lambda}{2} 4 - \binom{\lambda}{1} 2 + \frac{19}{5} \right] \right. \right.$$

$$\left. + \bar{c}^{(3-\lambda)} \left[\binom{\lambda}{4} 24 + \binom{\lambda}{3} 36 - \binom{\lambda}{2} 6 - \binom{\lambda}{1} 9 + 9 \right] \right]$$

$$\left. - \sim \right\}$$

$$+ n^{(8)} B(c-2, n-8) q^4 - n^{(7)} B(c-2, n-7) q^3 (\bar{c}\,4 - 16) + n^{(6)} B(c-2, n-6) q^2 \left(\bar{c}^{(2)} 6 - \bar{c}\,38 + \frac{208}{3}\right)$$

$$- n^{(5)} B(c-2, n-5) q \left(\bar{c}^{(3)} 4 - \bar{c}^{(2)} 28 + \bar{c}\,76 - \frac{384}{5}\right) + n^{(4)} B(c-2, n-4) \left(\bar{c}^{(4)} - \bar{c}^{(3)} 6 + \bar{c}^{(2)} 15 - \bar{c}\,15\right)$$

$$+ n^{(8)} B(c-6, n-8) p^4 - n^{(7)} B(c-5, n-7) p^3 (c\,4 - 16) + n^{(6)} B(c-4, n-6) p^2 \left(c^{(2)} 6 - c\,38 + \frac{208}{3}\right)$$

$$- n^{(5)} B(c-3, n-5) p \left(c^{(3)} 4 - c^{(2)} 28 + c\,76 - \frac{384}{5}\right) + n^{(4)} B(c-2, n-4) \left(c^{(4)} - c^{(3)} 6 + c^{(2)} 15 - c\,15\right) \quad \cdots\cdots (4.12)$$

次に (4.11) の { } 内の $[(n-d)^{(2)} - (n-d)\bar{c}]$ を因子とし他の因子は d のみの項とその対称の項とを集め

$$\sum_{d=0}^{c} \binom{n}{d} p^d q^{n-d} \left\{ \frac{4}{p^3 q} \left[d^{(6)} - d^{(5)} (c\,3 - 8) + d^{(4)} (c^{(2)} 3 - c\,11 + 12) - d^{(3)} (c^{(3)} - c^{(2)} 3 + c\,3) \right] \right.$$

$$\left. \cdot \left[(n-d)^{(2)} - (n-d)\bar{c} \right] + \sim \right\}$$

$$= 4 \left\{ n^{(8)} B(c-2, n-8) p q^3 - n^{(7)} B(c-2, n-7) p q^2 (\bar{c}\,3 - 8) + n^{(6)} B(c-2, n-6) p q (\bar{c}^{(2)} 3 - \bar{c}\,11 + 12) \right.$$

$$- n^{(5)} B(c-2, n-5) p (\bar{c}^{(3)} - \bar{c}^{(2)} 3 + \bar{c}\,3)$$

$$+ n^{(8)} B(c-6, n-8) p^3 q - n^{(7)} B(c-5, n-7) p^2 q (c\,3 - 8) + n^{(6)} B(c-4, n-6) p q (c^{(2)} 3 - c\,11 + 12)$$

$$- n^{(5)} B(c-3, n-5) q (c^{(3)} - c^{(2)} 3 + c\,3)$$

$$- c \left[n^{(7)} B(c-1, n-7) q^3 - n^{(6)} B(c-1, n-6) q^2 (\bar{c}\,3 - 8) + n^{(5)} B(c-1, n-5) q (\bar{c}^{(2)} 3 - \bar{c}\,11 + 12) \right.$$

$$\left. - n^{(4)} B(c-1, n-4) (\bar{c}^{(3)} - \bar{c}^{(2)} 3 + \bar{c}\,3) \right]$$

$$- \bar{c} \left[n^{(7)} B(c-6, n-7) p^3 - n^{(6)} B(c-5, n-6) p^2 (c\,3 - 8) + n^{(5)} B(c-4, n-5) p (c^{(2)} 3 - c\,11 + 12) \right.$$

$$\left. \left. - n^{(4)} B(c-3, n-4) (c^{(3)} - c^{(2)} 3 + c\,3) \right] \right\} \quad \cdots (4.13)$$

右項の $c-\lambda$ を一つだけ中の方に寄せると

$$n^{(8)}b(c-2,n-8)pq^3 - n^{(7)}b(c-2,n-7)pq^2(\bar{c}3-8) + n^{(6)}b(c-2,n-6)pq(\bar{c}^{(2)}3-\bar{c}11+12)$$

$$-n^{(5)}b(c-2,n-5)p(\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}3) \qquad + \sim$$

$$-c\,[n^{(7)}b(c-1,n-7)q^3 - n^{(6)}b(c-1,n-6)q^2(\bar{c}3-8) + n^{(5)}b(c-1,n-5)q(\bar{c}^{(2)}3-\bar{c}11+12)$$

$$-n^{(4)}b(c-1,n-4)(\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}3)] \qquad - \sim$$

$$= nb(c,n-1)\left\{\frac{c^{(2)}-c\cdot c}{pq^2}[\bar{c}^{(5)}-\bar{c}^{(4)}(\bar{c}3-8)+\bar{c}^{(3)}(\bar{c}^{(2)}3-\bar{c}11+12)-\bar{c}^{(2)}(\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}3)]\right.$$

$$\left. + \sim \right\}$$

$$= 0$$

となって，(4.13) は次のようになる。

$$(4.13) = 4\left\{ n^{(8)}B(c-3,n-8)pq^3 - n^{(7)}B(c-3,n-7)pq^2(\bar{c}3-8)\right.$$

$$+ n^{(6)}B(c-3,n-6)pq(\bar{c}^{(2)}3-\bar{c}11+12) - n^{(5)}B(c-3,n-5)p(\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}3)$$

$$+ n^{(8)}B(c-5,n-8)p^3q - n^{(7)}B(c-4,n-7)p^2q(c3-8)$$

$$+ n^{(6)}B(c-3,n-6)pq(c^{(2)}3-c11+12) - n^{(5)}B(c-2,n-5)q(c^{(3)}-c^{(2)}3+c3)$$

$$-c\,[n^{(7)}B(c-2,n-7)q^3 - n^{(6)}B(c-2,n-6)q^2(\bar{c}3-8) + n^{(5)}B(c-2,n-5)q(\bar{c}^{(2)}3-\bar{c}11+12)$$

$$-n^{(4)}B(c-2,n-4)(\bar{c}^{(3)}-\bar{c}^{(2)}3+\bar{c}3)]$$

$$-\bar{c}\,[n^{(7)}B(c-5,n-7)p^3 - n^{(6)}B(c-4,n-6)p^2(c3-8) + n^{(5)}B(c-3,n-5)p(c^{(2)}3-c11+12)$$

$$\left. - n^{(4)}B(c-2,n-4)(c^{(3)}-c^{(2)}3+c3)] \right\} \qquad \cdots (4.14)$$

(4.11) の残りの項について計算すると

$$\sum_{d=0}^{c}\binom{n}{d}p^d q^{n-d}\left\{ -\frac{2}{p^2q^2}[d^{(3)}+d^{(2)}3][(n-d)^{(4)}-(n-d)^{(3)}(\bar{c}2-3)+(n-d)^{(2)}(\bar{c}^{(2)}-\bar{c}+1)]\right.$$

$$- \sim$$

$$+\frac{6}{p^2q^2}[d^{(4)}-d^{(3)}(c2-3)+d^{(2)}(c^{(2)}-c+1)][(n-d)^{(4)}-(n-d)^{(3)}(\bar{c}2-3)+(n-d)^{(2)}(\bar{c}^{(2)}-\bar{c}+1)]$$

$$\left. + \frac{2}{3}\frac{1}{p^2q^2}[d^{(3)}+d^{(2)}3][(n-d)^{(3)}+(n-d)^{(2)}3] \right\}$$

$$= -2\{ n^{(7)}B(c-3,n-7)pq^2 - n^{(6)}B(c-3,n-6)pq(\bar{c}2-3) + n^{(5)}B(c-3,n-5)p(\bar{c}^{(2)}-\bar{c}+1)$$

$$+ n^{(7)}B(c-4,n-7)p^2q - n^{(6)}B(c-3,n-6)pq(c2-3) + n^{(5)}B(c-2,n-5)q(c^{(2)}-c+1)\}$$

$$-6\{n^{(6)}B(c-2,n-6)q^2 - n^{(5)}B(c-2,n-5)q(\bar{c}2-3) + n^{(4)}B(c-2,n-4)(\bar{c}^{(2)}-\bar{c}+1)$$

$$+n^{(6)}B(c-4,n-6)p^2 - n^{(5)}B(c-3,n-5)p(c2-3) + n^{(4)}B(c-2,n-4)(c^{(2)}-c+1)\}$$

$$+6\{n^{(8)}B(c-4,n-8)p^2q^2 - n^{(7)}B(c-3,n-7)pq^2(c2-3) - n^{(7)}B(c-4,n-7)p^2q(\bar{c}2-3)$$

$$+n^{(6)}B(c-2,n-6)q^2(c^{(2)}-c+1) + n^{(6)}B(c-3,n-6)pq(c2-3)(\bar{c}2-3) + n^{(6)}B(c-4,n-6)p^2(\bar{c}^{(2)}-\bar{c}+1)$$

$$-n^{(5)}B(c-2,n-5)q(c^{(2)}-c+1)(\bar{c}2-3) - n^{(5)}B(c-3,n-5)p(\bar{c}^{(2)}-\bar{c}+1)(c2-3)$$

$$+n^{(4)}B(c-2,n-4)(c^{(2)}-c+1)(\bar{c}^{(2)}-\bar{c}+1)\}$$

$$+n^{(6)}B(c-3,n-6)pq\frac{2}{3} + n^{(5)}B(c-2,n-5)q2 + n^{(5)}B(c-3,n-5)p2$$

$$+n^{(4)}B(c-2,n-4)6 \qquad \cdots (4.15)$$

(4.12), (4.14), (4.15) を加え

$$A_4 = nb(c,n-1)A_{4,0} + B(c-2,n-4)\{n^{(8)} - n^{(7)}[(n-1)4-16]$$

$$+ n^{(6)}[(n-1)^{(2)}6-(n-1)38+\frac{208}{3}] - n^{(5)}[(n-1)^{(3)}4-(n-1)^{(2)}28+(n-1)76-\frac{384}{5}]$$

$$+ n^{(4)}[(n-1)^{(4)}-(n-1)^{(3)}6+(n-1)^{(2)}15-(n-1)15]\}$$

$B(c-2,n-4)$ の係数は $n^{(6)}\frac{1}{3} + n^{(5)}\frac{19}{5} + n^{(4)}9 = -f_4(n)+\frac{4!}{2!(2!)^2}(f_2(n))^2$ $\cdots(4.16)$

となり

$$A_4 = nb(c,n-1)A_{4,0} - B(c-2,n-4)\{f_4(n) - \frac{4!}{2!(2!)^2}(f_2(n))^2\} \qquad \cdots (4.17)$$

$$A_{4,0} = \frac{1}{q^3}[\bar{c}^{(5)}\frac{1}{3} + \bar{c}^{(4)}\frac{19}{5}+\bar{c}^{(3)}9] - \frac{1}{p^3}[c^{(5)}\frac{1}{3} + c^{(4)}\frac{19}{5}+c^{(3)}9]$$

$$+ \frac{c}{pq^2}[\bar{c}^{(4)}\frac{1}{3}+\bar{c}^{(3)}\frac{9}{5}] - \frac{\bar{c}}{p^2q}[c^{(4)}\frac{1}{3} + c^{(3)}\frac{9}{5}] \cdots (4.18)$$

$$=((\bar{c}^2+\bar{c}2)(\bar{c}-\frac{13}{5})+\frac{24}{5})\frac{1}{q^2}[\frac{\bar{c}^2+\bar{c}2}{3q}+\frac{c\bar{c}}{3p}] - \sim$$

よって(4.9)によって

$$K_{2m} = n b(c, n-1)\left\{ \frac{F_{2m}}{n^{(2)}} A_{0,0} + \frac{F_{2m} - \binom{2m}{2} F_{2m-2}\, f_2(n)}{n^{(4)}} A_{0,1} \right.$$

$$+ \frac{F_{2m} - \binom{2m}{2} F_{2m-2}\, f_2(n) - \binom{2m}{4} F_{2m-4}\left[f_4(n) - \frac{4!}{2!(2!)^2}(f_2(n))^2 \right]}{n^{(6)}} A_{0,2}$$

$$\left. + \binom{2m}{2} F_{2m-2} A_{2,0} + \binom{2m}{4} F_{2m-4} A_{4,0} \right\}$$

$$+ \left\{ F_{2m} - \binom{2m}{2} F_{2m-2}\, f_2(n) - \binom{2m}{4} F_{2m-4}\left[f_4(n) - \frac{4!}{2!(2!)^2}(f_2(n))^2 \right] \right\} \cdot$$

$$\cdot B(c-3, n-6)$$

$$+ \sum_{t=3}^{m} F_{2m-2t} \binom{2m}{2t} A_{2t} \qquad \cdots (4\cdot 19)$$

よって K_4 は次のようになる.

$$K_4 = n b(c, n-1)\left\{ \frac{n+1}{3}\left[n(n^2-1) + (n^2-1)\frac{18}{5} - \frac{24}{5} \right] A_{0,0} \right.$$

$$- \frac{n+1}{3}\left[n+1 + \frac{2}{5} \right] A_{0,1}$$

$$\left. + 2n(n^2-1) A_{2,0} + A_{4,0} \right\} \qquad \cdots (4\cdot 20)$$

(4.20)は筆者が別に求めたもの[4]と一致している。

むすび

できるだけ一般的に計算を進めたいと思っ

で工夫もしてみたが，今はこの程度にとまった。なお筆者が前に書いたもの[4]とは記号も計算の運び方もちがっているので，それを読まれるときは注意してほしい。

電子計算機による数値例については，戸田英雄君が行っている。

参考文献

(1) 山内：統計的品質管理（電気学会編，昭和28年5月）第6章抜取検査．p. 291.

(2) M.E. Wise: " A Quickly Convergent Expansion for Cumulative Hypergeometric Probabilities, Direct and Inverse," Biometrika, vol. 41 (1954) pp. 317~329

(3) G.J. Lieberman and ~~Donald~~ B. Owen: Tables of the Hypergeometric Probability Distribution (1961) (Stanford Univ. Press) pp. 16~22

(4) 山内：本文の同じ題目で近刊"応用統計学"に投稿中.

附録 1　$f_{2r}(n)$ の値

$$f_2 = (n+1)^{(3)}/3$$

$$f_4 = \frac{3!}{5}\left\{(n+2)^{(5)} + (n+1)^{(3)}\frac{5}{3}\right\}$$

$$f_6 = \frac{5!}{7}\left\{(n+3)^{(7)} + (n+2)^{(5)}7 + (n+1)^{(3)}\frac{7}{3}\right\}$$

$$f_8 = \frac{7!}{9}\left\{(n+4)^{(9)} + (n+3)^{(7)}9\cdot 2 + (n+2)^{(5)}\frac{9\cdot 21}{5} + (n+1)^{(3)}\frac{9}{3}\right\}$$

$$f_{10} = \frac{9!}{11}\left\{(n+5)^{(11)} + (n+4)^{(9)}\frac{11\cdot 10}{3} + (n+3)^{(7)}11\cdot 21 + (n+2)^{(5)}11\cdot 17 + (n+1)^{(3)}\frac{11}{3}\right.$$

$$f_{12} = \frac{11!}{13}\left\{(n+6)^{(13)} + (n+5)^{(11)}13\cdot 5 + (n+4)^{(9)}\frac{13\cdot 11\cdot 19}{3} + (n+3)^{(7)}\frac{13\cdot 11\cdot 128}{7}\right.$$
$$\left. + (n+2)^{(5)}\frac{13\cdot 11\cdot 31}{5} + (n+1)^{(3)}\frac{13}{3}\right\}$$

附録 2　F_{2m} の値

$$F_2 = (n+1)^{(3)}/3$$

$$F_4 = (n+3)^{(6)}/3 + (n+3)^{(5)}/5$$

$$F_6 = (n+5)^{(9)}5/3^2 + (n+5)^{(8)} + (n+5)^{(7)}/7$$

$$F_8 = (n+7)^{(12)}35/3^3 + (n+7)^{(11)}14/3 + (n+7)^{(10)}41/15$$
$$+ (n+7)^{(9)}/9$$

$$F_{10} = (n+9)^{(15)}35/3^2 + (n+9)^{(14)}70/3 + (n+9)^{(13)}31$$
$$+ (n+9)^{(12)}23/3 + (n+9)^{(11)}/11$$

$$F_{12} = (n+11)^{(18)}11\cdot 7\cdot 5/3^3 + (n+11)^{(17)}11\cdot 7\cdot 5/3 + (n+11)^{(16)}11\cdot 83/3$$
$$+ (n+11)^{(15)}11\cdot 268/15 + (n+11)^{(14)}157/7$$
$$+ (n+11)^{(13)}/13$$

超幾何分布における山内の式

[記号]

$$p_d = \frac{\binom{Np_0}{d}\binom{N-Np_0}{n-d}}{\binom{N}{n}}, \quad (d=0,1,\cdots,n) \qquad (1)$$

$$0 < Np_0 < N$$

$d=0,1,\cdots,n$ なる値がそれぞれ 確率 $p_0, p_1, \cdots, p_n$ で出現する.

$$P = \sum_{d=0}^{c} p_d \equiv H(N, n, Np_0, c) \qquad (2)$$

$$p_d \sim \binom{n}{d} p_0^d (1-p_0)^{n-d} \equiv b(d, n) \qquad (3)$$

$$P \sim \sum_{d=0}^{c} \binom{n}{d} p_0^d (1-p_0)^{n-d} \equiv B(c, n, p_0) \qquad (4)$$

[山内の近似式(1953)]

$$\left.\begin{aligned} P &= B(c,n,p_0) - n\binom{n-1}{c} p_0^c (1-p_0)^{n-c-1} \cdot \left[\frac{A_1}{N} + \frac{A_2}{N^2} - \frac{A_1^3}{N^2 p_0(1-p_0)}\right] \\ A_1 &= -\left(c-(n-1)p_0\right)/2 \\ A_2 &= A_1^2\left[\frac{1}{6p_0(1-p_0)} + \frac{7}{6p_0}\right] + A_1\left[\frac{3c}{4p_0} + \frac{1}{6(1-p_0)} + \frac{1}{6}\right] \\ &\quad + \frac{c}{6} - \frac{c}{12p_0} \end{aligned}\right\} \qquad (5)$$

実用的に $H(N, n, Np_0, c)$ を $B(c, n, p_0)$ で近似するときの目安として $\frac{1}{N}$ のオーダの項：

$$\frac{n}{N}\,\frac{c-(n-1)p_0}{2}\binom{n-1}{c} p_0^c (1-p_0)^{n-1-c}$$

が近似の尺度となる

[Wise－山内の式 (1971)]

$$p = \frac{Np_0 - \frac{c}{2}}{N - \frac{n-1}{2}} \quad , \left(N - \frac{n-1}{2} = M \quad \text{とおく} \right)$$

なる p を導入して M^{-2} の級数展開が得られた。

$$\underline{P} = B(c, n, p)$$

$$+ \frac{n \cdot b(c, n-1, p)}{2!\ (2M)^2} \left\{ \frac{n+1}{3} \left[(n-1-c)q - cp \right] - \frac{(n-1-c)^2 + 2(n-1-c)}{3q} + \frac{c^2+2c}{3p} \right\}$$

Wise

山内

$$+ \frac{n \cdot b(c, n-1, p)}{4!\ (2M)^4} \left\{ \frac{n+1}{3} \left[(n-1-c)q - cp \right] \cdot \left[n^3 - n + \frac{(n^2-1)18 - 24}{5} \right] \right.$$

$$+ \left[- \frac{(n-1-c)^2 + 2(n-1-c)}{3q} + \frac{c^2+2c}{3p} \right] 2(n^3 - n)$$

$$+ (n+1+\tfrac{2}{5})(n+1) \cdot \left[\frac{c(n-1-c)}{3q} \cdot (c-1) - \frac{c(n-1-c)}{3p} \cdot (n-1-c-1) \right]$$

$$+ \left(\left[(n-1-c)^2 + 2(n-1-c) \right] (n-1-c-\tfrac{13}{5}) + \tfrac{24}{5} \right) \frac{1}{q^2} \left[\frac{(n-1-c)^2 + 2(n-1-c)}{3q} + \frac{c(n-1-c)}{3p} \right]$$

$$\left. - \left(\left[c^2 + 2c \right] (c - \tfrac{13}{5}) + \tfrac{24}{5} \right) \frac{1}{p^2} \left[\frac{c^2+2c}{3p} + \frac{c(n-1-c)}{3q} \right] \right\}$$

$$+ \cdots$$

N	項数	exp(lnΓ(N+1))	秒	N!	秒	(Γ(N+1)−N!)/N!
2	25	.2000001123195979D+01	.11	.2000000000000000D+01	.00	.5615976740978831D-06
3	23	.6000000016053141D+01	.10	.6000000000000000D+01	.00	.2675523423541415D-08
4	15	.2399999999999564D+02	.08	.2400000000000000D+02	.01	-.1817834077711186D-12
5	13	.1200000000000065D+03	.07	.1200000000000000D+03	.01	.5391057970408699D-13
6	11	.7199999999999560D+03	.07	.7200000000000000D+03	.02	-.6112394541131097D-13
7	11	.5039999999999949D+04	.07	.5040000000000000D+04	.02	-.1003289162507302D-13
8	11	.4031999999999993D+05	.07	.4032000000000000D+05	.01	-.1741111664015328D-14
9	9	.3628800000000069D+06	.07	.3628800000000000D+06	.02	.1895281893410150D-13
10	9	.3628800000000024D+07	.07	.3628800000000000D+07	.02	.6611681611764134D-14
11	9	.3991680000000009D+08	.07	.3991680000000000D+08	.02	.2235458357893051D-14
12	9	.4790016000000002D+09	.07	.4790016000000000D+09	.02	.3825409279543019D-15
13	9	.6227020799999996D+10	.07	.6227020800000000D+10	.02	-.5928617880601521D-15
14	9	.8717829119999989D+11	.07	.8717829120000000D+11	.02	-.1258709588373926D-14
15	9	.1307674368000002D+13	.07	.1307674368000000D+13	.03	.1899801399363573D-14
16	9	.2092278988800003D+14	.07	.2092278988800000D+14	.03	.1669345644200852D-14
17	7	.3556874280959990D+15	.07	.3556874280960000D+15	.03	-.2837540109732523D-14
18	7	.6402373705727990D+16	.07	.6402373705728000D+16	.03	-.1502128420182006D-14
19	7	.1216451004088319D+18	.06	.1216451004088320D+18	.03	-.8318255290178001D-15
20	7	.2432902008176639D+19	.06	.2432902008176640D+19	.04	-.5425619262772061D-15
21	7	.5109094217170941D+20	.06	.5109094217170944D+20	.04	-.5323839969242426D-15
22	7	.1124000727777607D+22	.07	.1124000727777608D+22	.03	-.6472878371160204D-15
23	7	.2585201673888495D+23	.07	.2585201673888498D+23	.04	-.8663512880336322D-15
24	7	.6204484017332387D+24	.07	.6204484017332394D+24	.04	-.1170345314729657D-14
25	7	.1551121004333096D+26	.07	.1551121004333099D+26	.04	-.1515888712635254D-14
26	7	.4032914611266049D+27	.06	.4032914611266056D+27	.05	-.1846406449727027D-14
27	7	.1088886945041833D+29	.07	.1088886945041835D+29	.04	-.2219099615191703D-14
28	7	.3048883446117130D+30	.07	.3048883446117139D+30	.04	-.2673152347384450D-14
29	7	.8841761993739675D+31	.07	.8841761993739702D+31	.04	-.3042204990471046D-14
30	7	.2652528598121901D+33	.07	.2652528598121911D+33	.05	-.3461494721226616D-14
31	7	.8222838654177953D+34	.07	.8222838654177923D+34	.05	.3644356064957852D-14
32	7	.2631308369336944D+36	.06	.2631308369336933D+36	.06	.3452663155018681D-14
33	7	.8683317618811915D+37	.07	.8683317618811887D+37	.05	.3225353536444924D-14
34	7	.2952327990396050D+39	.07	.2952327990396041D+39	.05	.3008634621650236D-14
35	7	.1033314796638617D+41	.06	.1033314796638614D+41	.06	.2735941532359694D-14
36	7	.3719933267899021D+42	.06	.3719933267899012D+42	.07	.2486142049874794D-14
37	7	.1376375309122638D+44	.07	.1376375309122635D+44	.06	.2179689140070408D-14
38	7	.5230226174666021D+45	.06	.5230226174666011D+45	.07	.1908309742910059D-14
39	7	.2039788208119748D+47	.07	.2039788208119744D+47	.06	.1619199489273398D-14
40	7	.8159152832478988D+48	.06	.8159152832478977D+48	.07	.1273996839536251D-14
41	7	.3345252661316384D+50	.06	.3345252661316381D+50	.07	.1030110948422173D-14
42	7	.1405006117752881D+52	.06	.1405006117752880D+52	.08	.6194697303552833D-15
43	7	.6041526306337385D+53	.07	.6041526306337384D+53	.07	.2867073888580512D-15
44	7	.2658271574788449D+55	.06	.2658271574788449D+55	.08	-.5850405841969392D-16
45	7	.1196222208654801D+57	.07	.1196222208654802D+57	.07	-.3975386883234512D-15
46	7	.5502622159812085D+58	.07	.5502622159812089D+58	.07	-.8088677052610630D-15
47	7	.2586232415111679D+60	.07	.2586232415111682D+60	.07	-.1184970925726716D-14
48	7	.1241391559253605D+62	.07	.1241391559253607D+62	.08	-.1489690566850679D-14
49	7	.6082818640342664D+63	.06	.6082818640342676D+63	.08	-.1858903004352270D-14
50	7	.3041409320171331D+65	.07	.3041409320171338D+65	.08	-.2319329052372524D-14
51	7	.1551118753287378D+67	.06	.1551118753287382D+67	.09	-.2685810102148830D-14
52	7	.8065817517094363D+68	.06	.8065817517094388D+68	.09	-.3053966091350860D-14
53	7	.4274883284060010D+70	.06	.4274883284060026D+70	.09	-.3575780523895119D-14
54	7	.2308436973392405D+72	.06	.2308436973392414D+72	.09	-.4026354439165213D-14
55	7	.1269640335365822D+74	.07	.1269640335365828D+74	.09	-.4332759317942727D-14

図-1 exp(lnΓ(N+1)) とN!の比較

N	項数	(1) $ln\,\Gamma(N+1)$	秒	(2) $ln N + ln(N-1) + \cdots + ln 2$	秒	(1) − (2)	$ln(N!)$	秒
2	25	.6931477421577771D+00	.10	.6931471805599454D+00	.02	.5615978316765753D-06	.6931471805599454D+00	.01
3	23	.1791759471903578D+01	.08	.1791759469228055D+01	.03	.2675523329796758D-08	.1791759469228055D+01	.02
4	15	.3178053830347764D+01	.07	.3178053830347946D+01	.04	-.1819985134821067D-12	.3178053830347946D+01	.02
5	13	.4787491742782100D+01	.07	.4787491742782046D+01	.05	.5365152766501069D-13	.4787491742782046D+01	.02
6	11	.6579251212010040D+01	.06	.6579251212010101D+01	.07	-.6136063879225162D-13	.6579251212010101D+01	.02
7	11	.8525161361065404D+01	.05	.8525161361065414D+01	.08	-.1019323514483972D-13	.8525161361065414D+01	.02
8	11	.1060460290274525D+02	.06	.1060460290274525D+02	.09	-.2019218126037003D-14	.1060460290274525D+02	.03
9	9	.1280182748008149D+02	.05	.1280182748008147D+02	.10	.1859623566247137D-13	.1280182748008147D+02	.03
10	9	.1510441257307552D+02	.05	.1510441257307552D+02	.12	.6210310043996969D-14	.1510441257307552D+02	.03
11	9	.1750230784587389D+02	.06	.1750230784587389D+02	.13	.1831867990631508D-14	.1750230784587389D+02	.03
12	9	.1998721449566189D+02	.06	.1998721449566189D+02	.14	.0000000000000000D+00	.1998721449566189D+02	.03
13	9	.2255216385312342D+02	.05	.2255216385312342D+02	.15	-.9159339953157541D-15	.2255216385312342D+02	.04
14	9	.2519122118273868D+02	.05	.2519122118273868D+02	.16	-.1498801083243961D-14	.2519122118273868D+02	.04
15	9	.2789927138384089D+02	.05	.2789927138384089D+02	.18	.1748601263784622D-14	.2789927138384089D+02	.04
16	9	.3067186010608067D+02	.06	.3067186010608067D+02	.19	.1401656568589260D-14	.3067186010608067D+02	.04
17	7	.3350507345013689D+02	.05	.3350507345013689D+02	.20	-.3191891195797325D-14	.3350507345013689D+02	.04
18	7	.3639544520803305D+02	.05	.3639544520803305D+02	.21	-.1915134717478395D-14	.3639544520803305D+02	.05
19	7	.3933988418719949D+02	.05	.3933988418719949D+02	.23	-.1304512053934559D-14	.3933988418719949D+02	.04
20	7	.4233561646075348D+02	.04	.4233561646075349D+02	.24	-.1054711873393899D-14	.4233561646075349D+02	.04
21	7	.4538013889847691D+02	.05	.4538013889847691D+02	.25	-.1054711873393899D-14	.4538013889847691D+02	.04
22	7	.4847118135183522D+02	.05	.4847118135183522D+02	.26	-.1165734175856414D-14	.4847118135183522D+02	.05
23	7	.5160667556776437D+02	.05	.5160667556776437D+02	.28	-.1360023205165817D-14	.5160667556776437D+02	.05
24	7	.5478472939811232D+02	.05	.5478472939811232D+02	.29	-.1609823385706477D-14	.5478472939811232D+02	.05
25	7	.5800360522298052D+02	.05	.5800360522298052D+02	.30	-.1915134717478395D-14	.5800360522298052D+02	.06
26	7	.6126170176100200D+02	.05	.6126170176100200D+02	.31	-.2192690473634684D-14	.6126170176100200D+02	.05
27	7	.6455753862700633D+02	.05	.6455753862700633D+02	.32	-.2442490654175344D-14	.6455753862700633D+02	.06
28	7	.6788974313718153D+02	.05	.6788974313718154D+02	.34	-.2775557561562891D-14	.6788974313718153D+02	.06
29	7	.7125703896716801D+02	.05	.7125703896716801D+02	.35	-.3053113317719180D-14	.7125703896716801D+02	.06
30	7	.7465823634883016D+02	.05	.7465823634883016D+02	.36	-.3330669073875470D-14	.7465823634883016D+02	.06
31	7	.7809222355331531D+02	.05	.7809222355331531D+02	.37	.3941291737419306D-14	.7809222355331531D+02	.06
32	7	.8155795945611504D+02	.05	.8155795945611504D+02	.39	.3608224830031759D-14	.8155795945611504D+02	.06
33	7	.8505446701758152D+02	.05	.8505446701758152D+02	.40	.3275157922644212D-14	.8505446701758152D+02	.07
34	7	.8858082754219768D+02	.05	.8858082754219768D+02	.42	.2997602166487923D-14	.8858082754219768D+02	.06
35	7	.9213617560368710D+02	.05	.9213617560368709D+02	.42	.2720046410331634D-14	.9213617560368709D+02	.08
36	7	.9571969454214321D+02	.05	.9571969454214320D+02	.44	.2386979502944087D-14	.9571969454214320D+02	.07
37	7	.9933061245478743D+02	.05	.9933061245478743D+02	.45	.1998401444325282D-14	.9933061245478743D+02	.07
38	7	.1029681986145138D+03	.05	.1029681986145138D+03	.46	.1665334536937735D-14	.1029681986145138D+03	.07
39	7	.1066317602606435D+03	.05	.1066317602606435D+03	.47	.1332267629550188D-14	.1066317602606435D+03	.08
40	7	.1103206397147574D+03	.05	.1103206397147574D+03	.49	.9992007221626409D-15	.1103206397147574D+03	.08
41	7	.1140342117814617D+03	.05	.1140342117814617D+03	.50	.7216449660063518D-15	.1140342117814617D+03	.08
42	7	.1177718813997451D+03	.05	.1177718813997451D+03	.51	.3885780586188048D-15	.1177718813997451D+03	.09
43	7	.1215330815154386D+03	.05	.1215330815154386D+03	.52	.0000000000000000D+00	.1215330815154386D+03	.08
44	7	.1253172711493569D+03	.05	.1253172711493569D+03	.53	-.2775557561562891D-15	.1253172711493569D+03	.09
45	7	.1291239336391272D+03	.05	.1291239336391272D+03	.55	-.6661338147750939D-15	.1291239336391272D+03	.09
46	7	.1329525750356163D+03	.05	.1329525750356163D+03	.56	-.9992007221626409D-15	.1329525750356163D+03	.09
47	7	.1368027226373264D+03	.05	.1368027226373264D+03	.57	-.1332267629550188D-14	.1368027226373264D+03	.10
48	7	.1406739236482343D+03	.05	.1406739236482343D+03	.58	-.1554312234475219D-14	.1406739236482343D+03	.09
49	7	.1445657439463449D+03	.05	.1445657439463449D+03	.59	-.1887379141862766D-14	.1445657439463449D+03	.10
50	7	.1484777669517730D+03	.05	.1484777669517730D+03	.61	-.2220446049250313D-14	.1484777669517730D+03	.10
51	7	.1524095925844974D+03	.05	.1524095925844974D+03	.63	-.2553512956637860D-14	.1524095925844974D+03	.09
52	7	.1563608363030788D+03	.05	.1563608363030788D+03	.63	-.2775557561562891D-14	.1563608363030788D+03	.10
53	7	.1603311282166309D+03	.05	.1603311282166309D+03	.65	-.3219646771412954D-14	.1603311282166309D+03	.10
54	7	.1643201122631952D+03	.05	.1643201122631952D+03	.66	-.3552713678800501D-14	.1643201122631952D+03	.11
55	7	.1683274454484277D+03	.05	.1683274454484277D+03	.67	-.3774758283725532D-14	.1683274454484277D+03	.10
[illegible]	7	.17225275[illegible]	.05	.17255276713[illegible]	[illegible]	.1410782519111307[illegible]	[illegible]	.10

57	7	.1763958484069973D+03	.05	.1763958484069974D+03	.70	-.4529869796038111D-14	.1683274454484277D+03	.10
58	7	.1804562914175438D+03	.05	.1804562914175438D+03	.71	-.4773959005888173D-14	.1683274454484277D+03	.10 → N! overflow
59	7	.1845338288614495D+03	.05	.1845338288614495D+03	.72	-.4884981308350689D-14	.1683274454484277D+03	.10
60	5	.1886281734236716D+03	.04	.1886281734236716D+03	.74	-.4996003610813204D-14	.1683274454484277D+03	.10
61	5	.1927390472878449D+03	.05	.1927390472878449D+03	.75	-.5440092820663267D-14	.1683274454484277D+03	.10
62	5	.1968661816728900D+03	.05	.1968661816728900D+03	.76	-.5662137425588298D-14	.1683274454484277D+03	.10
63	5	.2010093163992815D+03	.04	.2010093163992815D+03	.77	.9325873406851315D-14	.1683274454484277D+03	.10
64	5	.2051681994826412D+03	.05	.2051681994826412D+03	.78	.9214851104388799D-14	.1683274454484277D+03	.10
65	5	.2093425867525368D+03	.04	.2093425867525368D+03	.80	.8770761894538737D-14	.1683274454484277D+03	.10
66	5	.2135322414945633D+03	.04	.2135322414945633D+03	.81	.8437694987151190D-14	.1683274454484277D+03	.10
67	5	.2177369341139542D+03	.04	.2177369341139542D+03	.83	.8215650382226158D-14	.1683274454484277D+03	.10
68	5	.2219564418191303D+03	.05	.2219564418191303D+03	.83	.7993605777301127D-14	.1683274454484277D+03	.10
69	5	.2261905483237276D+03	.04	.2261905483237276D+03	.84	.7660538869913580D-14	.1683274454484277D+03	.10
70	5	.2304390435657770D+03	.04	.2304390435657770D+03	.86	.7438494264988549D-14	.1683274454484277D+03	.10
71	5	.2347017234428183D+03	.05	.2347017234428183D+03	.87	.7216449660063518D-14	.1683274454484277D+03	.10
72	5	.2389783895618343D+03	.05	.2389783895618343D+03	.88	.6883382752675971D-14	.1683274454484277D+03	.10
73	5	.2432688490029827D+03	.05	.2432688490029827D+03	.89	.6661338147750939D-14	.1683274454484277D+03	.10
74	5	.2475729140961869D+03	.04	.2475729140961869D+03	.91	.6217248937900877D-14	.1683274454484277D+03	.10
75	5	.2518904022097232D+03	.04	.2518904022097232D+03	.93	.5995204332975845D-14	.1683274454484277D+03	.10
76	5	.2562211355500095D+03	.05	.2562211355500095D+03	.93	.5551115123125783D-14	.1683274454484277D+03	.10
77	5	.2605649409718632D+03	.04	.2605649409718632D+03	.95	.5329070518200751D-14	.1683274454484277D+03	.10
78	5	.2649216497985528D+03	.04	.2649216497985528D+03	.96	.4884981308350689D-14	.1683274454484277D+03	.10
79	5	.2692910976510198D+03	.04	.2692910976510198D+03	.97	.4884981308350689D-14	.1683274454484277D+03	.10
80	5	.2736731242856937D+03	.05	.2736731242856937D+03	.98	.4218847493575595D-14	.1683274454484277D+03	.10
81	5	.2780675734403661D+03	.04	.2780675734403661D+03	1.00	.3996802888650564D-14	.1683274454484277D+03	.10
82	5	.2824742926876304D+03	.04	.2824742926876304D+03	1.01	.3996802888650564D-14	.1683274454484277D+03	.10
83	5	.2868931332954270D+03	.04	.2868931332954270D+03	1.02	.3774758283725532D-14	.1683274454484277D+03	.10
84	5	.2913239500942703D+03	.05	.2913239500942703D+03	1.03	.3108624468950438D-14	.1683274454484277D+03	.10
85	5	.2957666013507606D+03	.04	.2957666013507606D+03	1.05	.3108624468950438D-14	.1683274454484277D+03	.10
86	5	.3002209486470141D+03	.04	.3002209486470141D+03	1.06	.2664535259100376D-14	.1683274454484277D+03	.10
87	5	.3046868567656687D+03	.04	.3046868567656687D+03	1.07	.2220446049250313D-14	.1683274454484277D+03	.10
88	5	.3091641935801469D+03	.05	.3091641935801469D+03	1.08	.2220446049250313D-14	.1683274454484277D+03	.10
89	5	.3136528299498791D+03	.04	.3136528299498791D+03	1.10	.1776356839400250D-14	.1683274454484277D+03	.10
90	5	.3181526396202093D+03	.04	.3181526396202093D+03	1.11	.1554312234475219D-14	.1683274454484277D+03	.10
91	5	.3226634991267262D+03	.04	.3226634991267262D+03	1.12	.1332267629550188D-14	.1683274454484277D+03	.10
92	5	.3271852877037752D+03	.04	.3271852877037752D+03	1.13	.1110223024625157D-14	.1683274454484277D+03	.10
93	5	.3317178871969285D+03	.04	.3317178871969285D+03	1.15	.6661338147750939D-15	.1683274454484277D+03	.10
94	5	.3362611819791985D+03	.05	.3362611819791985D+03	1.16	.4440892098500626D-15	.1683274454484277D+03	.10
95	5	.3408150588707990D+03	.04	.3408150588707990D+03	1.17	.2220446049250313D-15	.1683274454484277D+03	.10
96	5	.3453794070622669D+03	.04	.3453794070622669D+03	1.19	-.2220446049250313D-15	.1683274454484277D+03	.10
97	5	.3499541180407702D+03	.05	.3499541180407702D+03	1.19	-.2220446049250313D-15	.1683274454484277D+03	.10
98	5	.3545390855194408D+03	.05	.3545390855194408D+03	1.20	-.4440892098500626D-15	.1683274454484277D+03	.10
99	5	.3591342053695754D+03	.05	.3591342053695754D+03	1.22	-.8881784197001252D-15	.1683274454484277D+03	.10
100	5	.3637393755555635D+03	.04	.3637393755555635D+03	1.23	-.1110223024625157D-14	.1683274454484277D+03	.10

図-2 ln(Γ(N+1)) と ln N! の比較

N= 100 SN= 40 C=, 16

NPO	P1	P2	P3
10.0	.9999996	.9999996	.9999995
15.0	.9999990	.9999990	.9999990
20.0	.9999747	1.0000023	.9999918
25.0	.9981362	.9989400	.9988530
30.0	.9723940	.9770684	.9770343
35.0	.8490185	.8572767	.8575598
40.0	.5808146	.5833520	.5834712
45.0	.2761124	.2698345	.2695708
50.0	.0832474	.0764569	.0763262

TIME= .7000

NPO	PE	ER
10.0	.9999980	.0000015
15.0	.9999995	-.0000005
20.0	.9999944	-.0000025
25.0	.9988522	.0000008
30.0	.9770260	.0000083
35.0	.8575714	-.0000116
40.0	.5834769	-.0000057
45.0	.2695580	.0000128
50.0	.0763256	.0000006

TIME= 1.0700

N= 200 SN= 80 C=, 32

NPO	P1	P2	P3
20.0	.9999997	.9999997	.9999996
30.0	.9999986	.9999986	.9999986
40.0	.9999979	.9999979	.9999978
50.0	.9999593	.9999890	.9999810
60.0	.9947863	.9962339	.9961442
70.0	.9054632	.9129043	.9130719
80.0	.5574017	.5591999	.5592844
90.0	.1627297	.1551904	.1549481
100.0	.0179172	.0149796	.0150384

TIME= .8400

NPO	PE	ER
20.0	.9999987	.0000010
30.0	.9999955	.0000031
40.0	.9999948	.0000031
50.0	.9999845	-.0000035
60.0	.9961402	.0000040
70.0	.9130753	-.0000034
80.0	.5592884	-.0000040
90.0	.1549392	.0000089
100.0	.0150432	-.0000048

TIME= 1.1400